

Element order versus minimal degree in permutation groups: an old lemma with new applications

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Abstract

In this note we present a simplified and slightly generalized version of a lemma the authors published in 1987. The lemma as stated here asserts that if the order of a permutation of n elements is greater than n^α then some non-identity power of the permutation has support size less than n/α . The original version made an unnecessary additional assumption on the cycle structure of the permutation; the proof of the present cleaner version follows the original proof verbatim. Application areas include parallel and sequential algorithms for permutation groups, the diameter of Cayley graphs of permutation groups, and the automorphisms of combinatorial structures with regularity constraints such as Latin squares, Steiner 2-designs, and strongly regular graphs. This note also serves as a modest tribute to the junior author whose untimely passing is deeply mourned.

1 The lemma

For a permutation π on the set K , the *support* of π is defined as the set $\text{supp}(\pi) = \{x \in K \mid x^\pi \neq x\}$. S_n denotes the symmetric group of degree n (and order $n!$). The *degree* of a permutation is the size of its support, and the *minimal degree* of a permutation group is the smallest degree of its non-identity elements. The following lemma shows that if there are elements of large order in a permutation group then its minimal degree is small.

Lemma. Let $\pi \in S_n$ have order $\geq n^\alpha$ for some $\alpha > 0$. Then $|\text{supp}(\pi^m)| \leq n/\alpha$ for some power $\pi^m \neq 1$.

Proof. Let π act on the set K where $|K| = n$. Let the order of π be $N = n^\alpha = \prod_{i=1}^r q_i$ where $q_i = p_i^{\beta_i} > 1$ are powers of distinct primes p_i . For each $x \in K$, let us consider the set $P(x)$ of those i for which q_i divides the length of the π -cycle through x . Clearly, for each $x \in K$,

$$\prod_{i \in P(x)} q_i \leq n. \quad (1)$$

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Let $n(i)$ denote the number of points $x \in K$ such that $i \in P(x)$. Let us estimate the weighted average W of the $n(i)$ with weights $\log q_i$. Recall that the sum of weights is $\sum \log q_i = \log N \geq \alpha \log n$, therefore (using Eq. (1))

$$W \leq \sum_{x \in K} \sum_{i \in P(x)} \log q_i / (\alpha \log n) \leq (n \log n) / (\alpha \log n) = n / \alpha. \quad (2)$$

Thus we infer that $n(i) \leq n / \alpha$ for some $i \leq r$. Now let $m = N / p_i$ be the corresponding maximal divisor of N . Clearly π^m is not the identity and it fixes all but $n(i)$ points. \square

2 History and applications

The original version of this lemma, published as [BS1, Lemma 3], assumed that the permutation π involved cycles of prime lengths p_i such that $\prod p_i \geq n^\alpha$. As we have seen, this assumption is unnecessary. The proof of the above cleaner form requires no new ideas, however; it is an essentially verbatim copy of the original proof.

Applications of this lemma are manifold, both old and new.

2.1 Degree of transitivity, diameter, parallel and sequential complexity of permutation groups

In [BS1], the authors used this lemma to obtain a short proof of a theorem of Jordan [Jo1, Jo2] on the degree of transitivity of permutation groups. In another simultaneous paper [BS2], the authors used this lemma to prove an $\exp(\sqrt{n \ln n}(1+o(1)))$ upper bound on the diameter of all Cayley graphs of S_n ; this bound was not improved until very recently [HS]. The breakthrough 2013 paper [HS] by Helfgott and Seress that reduced the diameter bound to quasipolynomial (exponential in a constant power of $\log n$) makes substantial use of a slight generalization of the lemma. In [BS3] the present authors, again using the Lemma, extended the $\exp(\sqrt{n \ln n}(1+o(1)))$ bound to all permutation groups of degree n (subgroups of S_n), which in this form is tight since S_n has cyclic subgroups of order $\exp(\sqrt{n \ln n}(1+o(1)))$ (product of small prime length cycles). In [BLS1], the Lemma played a key role in settling the parallel complexity of the permutation group membership problem; the subsequent papers [BLS2, BLS3] used the Lemma to design and analyze improved sequential algorithms for the same problem.

2.2 Latin squares

The present note was prompted by a conversation between Babai and Ian Wanless in July 2013 at a conference celebrating Peter Cameron at Queen Mary, University of London. Wanless mentioned the following recent result of his with Brendan McKay and Xiande Zhang. Recall that a *quasigroup* is a set with a binary operation such that all equations of the form $ax = b$ and $ya = b$ are uniquely solvable. In other words, a quasigroup is a set with a binary operation of which the multiplication table is a Latin square.

Theorem 2.1 (McKay, Wanless, Zhang [MWZ]). *No automorphism of a quasigroup of order n has order greater than $n^2/4$.*

We show that a slightly weaker bound, n^2 , is immediate from the Lemma. Indeed, let π be a non-identity automorphism of some quasigroup G of order n . Assume the order of π is greater than n^2 . Then some power $\pi^m \neq 1$ fixes more than half the elements of G . But the set of fixed points of a non-identity automorphism is a proper sub-quasigroup and therefore has order $\leq n/2$, a contradiction. \square

The McKay–Wanless–Zhang argument is not dissimilar to ours. They in fact prove the following stronger result.

Theorem 2.2 (McKay, Wanless, Zhang [MWZ]). *No autotopism of a quasigroup of order n has order greater than $n^2/4$.*

An *autotopism* of a quasigroup G is a triple (α, β, γ) of permutations of the set G such that for all $g, h \in G$ we have $\alpha(g)\beta(h) = \gamma(gh)$.

Wanless pointed out that a quadratic bound, $9n^2$, on the order of autotopisms of quasigroups of order n also follows quickly from the Lemma. Slightly modifying his argument, we infer a bound of $4n^2$ on the order of any autotopism. (Of course this is still a factor of 16 worse than their result.)

Indeed, assume $\theta = (\alpha, \beta, \gamma)$ is an autotopism of order greater than $(2n)^2$. We can view the autotopisms as acting on the union $G_1 \dot{\cup} G_2 \dot{\cup} G_3$ of three disjoint copies of G . In fact, the action on $U = G_1 \dot{\cup} G_2$ is faithful; let us consider this action. By the Lemma, some power $\theta^m \neq 1$ has more than n fixed points on U . Therefore it has at least one fixed point in each of G_1 and G_2 . A result by McKay, Meynert, and Myrvold [MMM] asserts that if a non-identity autotopism has a fixed point in G_1 and a fixed point in G_2 then the number of fixed points in each part is the same and that number cannot exceed $n/2$, a contradiction. \square

We do not expect the quadratic rate of growth in these results to be optimal. Let $f(n)$ denote the maximum of the orders of automorphisms of quasigroups of order $\leq n$. By the above, we have $f(n) = O(n^2)$.

Conjecture 2.1. $f(n) = o(n^2)$.

On the other hand, McKay et al. [MWZ] conjecture that 2 is the best possible exponent.

Conjecture 2.2 ([MWZ]). For every $\epsilon > 0$ and for infinitely many values of n ,

$$f(n) > n^{2-\epsilon}.$$

The construction of quasigroups with automorphisms of nearly quadratic order seems to face significant obstacles; currently, no superlinear examples are known (cf. [MWZ]).

It is noted in [MWZ] that the quadratic upper bound on the order of automorphisms holds in particular for Steiner Triple Systems (STSs) because such systems can be viewed as quasigroups: if x, y are distinct points of an STS then define xy as the third point in the unique triple containing x, y ; and set $xx = x$.

We observe that the quadratic upper bound for STSs extends to all Steiner 2-designs. A Steiner 2-design (also called a “regular linear space”) is an incidence geometry with lines of uniform length and exactly one line through every pair of points. If it has n points and each line has k points then this is a $2 - (n, k, 1)$ -design.

Proposition 2.3. *Let X be a Steiner 2-design with n points and with lines of length $k \geq 3$. Let m be the maximum order of automorphisms of X . Then*

$$(a) \ m < n^2$$

$$(b) \ m = O\left(n^{1+\frac{1}{k-2}}\right) \text{ where the implied constant is absolute.}$$

The proof follows by combining the Lemma with a bound on the number of fixed points of automorphisms of a Steiner 2-design by Davies [Da]. The details will appear in [Ba2].

2.3 Strongly regular graphs

In another paper [Ba1], the Lemma is used to establish strong structural constraints on the automorphism groups of strongly regular (SR) graphs.

Recall that a SR graph with parameters (n, k, λ, μ) has n vertices, is regular of degree k , each pair of adjacent vertices has λ common neighbors, and each pair of distinct, non-adjacent vertices has μ common neighbors. Disjoint unions of cliques of equal size and their complements are *trivial* examples of SR graphs. The line graphs of complete graphs and of complete bipartite graphs with equal parts are also SR; we refer to them and to their complements as *graphic* SR graphs.

The senior author has long suspected that the automorphism groups of non-trivial, non-graphic SR graphs are “small.” The following result gives a specific interpretation to this statement.

We say that a group H is *involved* in a group G if G has subgroups $L \triangleleft K \leq G$ such that $H \cong K/L$ (H is isomorphic to a quotient of a subgroup of G). We note that the automorphism groups of the trivial and the graphic SR graphs involve alternating groups of degree $\geq \sqrt{n}$. It turns out that in all other cases, the alternating groups involved are tiny. This has significant implications to attempts at subexponential and possibly even quasi-polynomial-time isomorphism tests for SR graphs, and it limits the possible primitive group actions on SR graphs. We state the result.

Theorem 2.4 ([Ba1]). *If the alternating group A_t is involved in the automorphism group of a non-trivial, non-graphic SR graph with n vertices then $t = O((\ln n)^2 / \ln \ln n)$.*

Theorem 2.4 is derived from a lemma that limits the number of fixed points of any non-identity automorphism of a regular graph in terms of its combinatorial and spectral parameters; from this, a bound on the orders of automorphisms follows via the Lemma above, and a bound on t is then immediate. To apply this general result to SR graphs, bounds on the second largest eigenvalue and on the parameters λ, μ are derived from known results.

3 Ákos Seress

(November 24, 1958 – February 13, 2013)

Apart from minor updates, this note was written in August 2013, about six months after the untimely passing of Ákos Seress [BO13]. I (Babai) had known Ákos, 8

years my junior, from his undergraduate years in the late 70s at Eötvös University, Budapest, where I was teaching at the time and he was a star student, already active in research. But our real meeting of minds occurred at a conference in Szeged, Hungary, in summer 1986, when by serendipity both of us missed the boat for a scenic afternoon ride. It was there, on the banks of the Tisza river, that I began to introduce Ákos, then a fresh Ph.D. in combinatorics, to asymptotic group theory and algorithmic group theory. That conversation evolved into a lifelong collaboration that produced 15 joint papers spanning a quarter century. I count several of our joint papers among the best of my career; this includes our last paper [BBS]. Ákos became a leader in algorithmic and computational group theory, an author of the definitive monograph on the subject [Ser1], a major contributor to the GAP symbolic algebra package, and a speaker at ICM 2006 in the algebra section.

Ákos was a most generous friend and colleague. He died at the age of 54 of renal cell carcinoma, a particularly aggressive form of cancer, diagnosed only six months earlier. The disease struck at the height of his creative powers, shortly after he had finished two breakthrough papers, one mathematical and one computational: the above-mentioned paper [HS] with Helfgott, a *tour de force* in the combinatorial theory of permutation groups, and a computational work on the Monster group [Ser2] that received the “Distinguished paper award” at the ISSAC 2012 conference and was hailed as “a groundbreaking work” that “marks a turning point in Majorana Theory.”

The Lemma discussed in this note was the fruit of the first hours of our collaboration, conceived even before the return of the boat, so I find it most appropriate to list Ákos as a coauthor.

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